THE INFLUENCE OF NON-LINEAR ELASTIC SYSTEMS ON THE MEASURING PRECISION OF MEASURING AND CONTROL SYSTEMS FOR MOMENTS/FORCES IN STATIC MODE

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Abstract – Using slides of building resilient systems of measurement and control of the moment/force in terms of large deformations may represent an alternative form in terms of simple construction, small volume of material and a high sensitivity of measurement systems. In this paper I will highlight the differences that arise between the characteristic elements of geometric equations for elastic deformation lamellae higher than those of order I.

Keywords - non-linear elastic, large displacement, deformed curve

1. Introduction

It is well known the fact that the majority of measuring and/or control systems have a linear characteristic between the variation of the input measure and the output measure and the variation of the input value is given by the following relation:

\[ X_e = S_a X_i \]  

Where: \( X_e \) – output measure; 
\( X_i \) – input measure; 
\( S_a \) – sensitivity of the measuring system.

We consider that from the theoretical point of view a measuring system of the torsion measuring / force, made up of 2–4 elastic slide elements that are exposed to deformations under the action of the torsion moment \( M_t \); the conditions of functioning and fixing of the slides is depicted schematically in figure 1.

Fig. 1. The functioning scheme: 1-elastic slide elements: 2-4 slides, 2-central axis, 3-supports, 4-poniters, 5-gradated scale
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The elastic slide elements are fixed at a head on the central axis (2) and the other head are fixed in the supports (3) that allow the displacement without friction on the longitudinal direction of the slide.

The pointer (4) visualizes the output measure of the system – αa angle. On the graduated scale (5), the αa angle is transformed in moment / force units.

In the current work, I will highlight the differences that appear between the characteristic elements of the geometrical elements of the elastic slide in case of great deformations differing from those of the 1st order.

For this case, the „small deformation hypothesis” can no longer be accepted and the exact geometrical shape of the deformed curve that the elastic elements under the action of the efforts must be taken into account as in fig. 2.

Figure 2. The geometrical elements of the exact deformed curves of an elastic slide

Due to the symmetrical distribution of the efforts on the entire system, we will carry out the analysis of the calculations of the deformations only for one slide.

Starting from the equation of the curve, we have:

\[
\frac{d\theta}{ds} = \frac{M(x,y)}{E \cdot I}
\]

(2)

Where M(x,y) is the bending moment in the x section, V the reaction in point 0, E is Young’s modulus, I is the moment of area of the beam. We will solve the equation (2) on the 2 intervals of the curve (1–2) as follows:

For the interval (0–1) we have

\[
\left(\frac{d\theta}{ds}\right)_{0-1} = \frac{-V \cdot x + M_0}{E \cdot I} = \frac{V}{E \cdot I} \cdot (m_0 - x)
\]

(3)

Mark down:

\[
\left(\frac{d\theta}{ds}\right)_{0-1} = k_\theta V \cdot (m_0 - x)
\]

We derive the relation (4) by the ds variable, taking into account the fact that

\[
\frac{dx}{ds} = \cos\theta \quad \text{and} \quad \frac{dy}{ds} = \sin\theta
\]

then

\[
\left(\frac{d^2\theta}{ds^2}\right)_{0-1} = -k_\theta^2 \cdot \cos\theta
\]

(5)

We multiply (5) with d\(\theta\) and by integration then

\[
\frac{1}{2} \cdot \left(\frac{d\theta}{ds}\right)^2_{0-1} = -k_\theta^2 \cdot \sin\theta + C_1
\]

(6)

For x=l and \(\theta=\alpha\) where \(\theta\) is the angle described by the tangent to the curve in the x section with the OX axis), and M (l_1) =0, so the integration constant will be

\[
C_1 = k_\theta^2 \cdot \sin\alpha
\]

Finally, we obtain the equation of the curve for the (0–1) interval

\[
\left(\frac{d\theta}{ds}\right)_{0-1} = \sqrt{2} \cdot k_\theta \cdot \sqrt{\sin\alpha - \sin\theta}
\]

(7)

For determining the equation of the exact curve on the (1–2) intervals, we proceed similarly and we have:

\[
\left(\frac{d\theta}{ds}\right)_{1-2} = \sqrt{2} \cdot k_\theta \cdot \sqrt{\sin\alpha - \sin\theta}
\]

(8)

From the equation of the curve over the two intervals, the following relation results:

\[
ds = \frac{1}{\sqrt{2} \cdot k_\theta} \cdot \left| \frac{d\theta}{\sqrt{\sin\alpha - \sin\theta}} \right|
\]

(9)

The three unknowns of the system are: reaction V, the α angle of inflection in point 1 and the β angle in point 2 where the known M\(_\alpha\) angle functions.

The first equation of the system results from the condition:

\[
M_\alpha = \frac{d\theta}{ds}(\theta = \beta) \cdot EI
\]

(10)
For \( x=1, \theta = \beta \) we have

\[
M_\alpha = \sqrt{2} \cdot k_V \cdot \sqrt{\sin \alpha - \sin \beta}
\]

and then,

\[
k_V = \frac{M_\alpha}{E \cdot l} \cdot \frac{1}{\sqrt{2} \cdot \sqrt{\sin \alpha - \sin \beta}} \tag{12}
\]

By introducing (12) in (9), we have

\[
ds = \frac{\sqrt{\sin \alpha - \sin \beta}}{M_\alpha/(E \cdot l)} \cdot \frac{d \theta}{\sqrt{\sin \alpha - \sin \beta}} \tag{13}
\]

In the calculation of the two rotation angles, we use two geometric conditions of the resting and the loading manner of the elastic element as it follows:

1) The projection of the deformed curve on the OX axis is constant:
\[
l_1 + l_2 = 1 \tag{14}
\]

2) The projection of the curve on the OY axis:
\[
y_1 + y_2 = 0 \tag{15}
\]

If \( dx = ds \cdot \cos \theta \) and \( dy = ds \cdot \sin \theta \) then:

\[
l_1 = \int_0^{l_1} ds = \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \int_0^\infty \frac{\cos \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} = \frac{2 \cdot \sqrt{\sin \alpha - \sin \beta}}{M_\alpha/(E \cdot l)} \cdot \sqrt{\sin \alpha}
\]

\[
l_2 = \int_0^{l_2} ds = \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \int_\beta^\infty \frac{\cos \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} = \frac{2 \cdot \sqrt{\sin \alpha - \sin \beta}}{M_\alpha/(E \cdot l)} \cdot \sqrt{\sin \alpha - \sin \beta} \tag{16}
\]

\[
l = l_1 + l_2 = \frac{2 \cdot \sqrt{\sin \alpha - \sin \beta}}{M_\alpha/(E \cdot l)} \cdot (\sqrt{\sin \alpha - \sin \beta} + \sqrt{\sin \alpha}) \tag{17}
\]

\[
y_1 = \int_0^{y_1} dy = \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \int_0^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} \tag{19}
\]

\[
y_2 = \int_0^{y_2} dy = \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \int_\beta^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} \tag{20}
\]

\[
y_1 + y_2 = \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \left( \int_0^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} + \int_\beta^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} \right) = 0 \tag{21}
\]

From (21) we have:

\[
\int_0^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} + \int_\beta^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} = 0 \tag{22}
\]

By resolving the system made up of transcendent equations (18) and (22), the two exact values of the \( \alpha \) and \( \beta \) are determined.

The maximum value of the \( y_{\text{max}} \) quota is determined for \( x=2l_1 \) by way of the relation (19):

\[
y_{\text{max}} = 2 \cdot y_1 = 2 \cdot \frac{\sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \int_0^\infty \frac{\sin \theta \cdot d \theta}{\sqrt{\sin \alpha - \sin \beta}} \tag{23}
\]

But \( l_1 = 2 \cdot l_1 \) by means of (16) resulting in the following:

\[
l_m = 2 \cdot l_1 = \frac{4 \cdot \sin \alpha - \sin \beta}{M_\alpha/(E \cdot l)} \cdot \sqrt{\sin \alpha} \tag{24}
\]
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In case of linear deformations of the 1st order the theory of small deformations can be applied: „The deformations of elastic bodies are small when compared to the dimensions of the bodies. This makes that calculation relations, when it comes to a square or a higher power of a deformation, these to be neglected in comparison with first order values“.

Concretely, this translates into the fact that:

\[ y'^2(x) = \tan^2(\theta) \cong 0 \] (25)

In these conditions resulting:

\[ \sin \theta \cong \tan \theta \cong \theta \quad \text{and} \quad \cos \theta \cong 1 \] (26)

and \[ dx = ds \cdot \cos(\theta) \cong ds, \quad dy = ds \cdot \sin(\theta) \cong ds \cdot \theta \] (27)

Applying relations 26 and 27 in 18 and 21, we obtain:

\[ l = \frac{\sqrt{\alpha_L - \beta_L}}{M_a/(E \cdot I)} \left( \int_0^{\alpha_L} \frac{d\theta_L}{\sqrt{\alpha_L - \theta_L} + \sqrt{\alpha_L - \beta_L}} + \int_{\beta_L}^{\alpha_L} \frac{d\theta_L}{\sqrt{\alpha_L - \beta_L}} \right) = \frac{2 \cdot \sqrt{\alpha_L - \beta_L}}{M_a/(E \cdot I)} \left( \sqrt{\alpha_L - \beta_L} + \sqrt{\alpha_L} \right) \] (28)

and

\[ \int_0^{\alpha_L} \frac{\theta_L \cdot d\theta_L}{\sqrt{\alpha_L - \theta_L} + \sqrt{\alpha_L - \beta_L}} + \int_{\beta_L}^{\alpha_L} \frac{\theta_L \cdot d\theta_L}{\sqrt{\alpha_L - \beta_L}} = 0 \] (29)

The notations \( \alpha_L \) and \( \beta_L \) correspond to the values of the two angles in the case of linear deformations of first order.

By integration on the two intervals, it results:

\[ M_a \cdot l = 2 \cdot \sqrt{\alpha_L - \beta_L} \cdot \left( \sqrt{\alpha_L - \beta_L} + \sqrt{\alpha_L} \right) \] (30)

and

\[ \sqrt{\alpha_L} \cdot 2 \cdot \alpha_L + \sqrt{\alpha_L - \beta_L} \cdot (\beta_L + 2 \cdot \alpha_L) = 0 \] (31)

From 31 results \( \beta_L = -\frac{3}{2} \cdot \alpha_L \) (32)

And from (30), (32) \( \alpha_L = \frac{M_a \cdot l}{12 \cdot E \cdot I} \) (33)

So, the rotation angle \( \beta_L \):

\[ |\beta_L| = \frac{M_a \cdot l}{4 \cdot E \cdot I} \] (34)

The maximum value \( y_{max} \) is calculated with:

\[ y_{max} = 2 \cdot y_{1L} = 2 \cdot \frac{\sqrt{\alpha_L - \beta_L}}{M_a/(E \cdot I)} \int_0^{\alpha_L} \frac{\theta_L \cdot d\theta_L}{\sqrt{\alpha_L - \theta_L}} = \frac{8}{3} \cdot \frac{\alpha_L}{M_a/(E \cdot I)} = \frac{1}{54} \frac{M_a \cdot l}{(E \cdot I)} \] (35)

And \( l_{\text{max}} \) with:

\[ l_{\text{max}} = 2 \cdot l_{1L} = 2 \cdot \frac{\sqrt{\alpha_L - \beta_L}}{M_a/(E \cdot I)} \int_0^{\alpha_L} \frac{d\theta_L}{\sqrt{\alpha_L - \theta_L}} = \frac{8}{3} \cdot \frac{\alpha_L}{M_a/(E \cdot I)} = \frac{2}{3} \cdot l \] (36)

If we use the notation: \( c_M = \frac{M_a \cdot l}{E \cdot I} \), we will obtain the following relations:

\[ c_M = 2 \cdot \sqrt{\sin \alpha - \sin \beta} \cdot \left( \sqrt{\sin \alpha - \sin \beta} + \sqrt{\sin \alpha} \right) \] (37)

\[ y_{\text{max}} = \frac{2}{c_M} \cdot \frac{\sin \alpha - \sin \beta}{\sin \alpha - \sin \beta} \cdot \int_0^{\infty} \frac{\sin \theta \cdot d\theta}{\sqrt{\sin \alpha - \sin \theta}} \cdot l \] (38)

\[ l_{\text{m}} = \frac{4 \cdot \sqrt{\sin \alpha - \sin \beta}}{c_M} \cdot \sqrt{\sin \alpha} \cdot l \] (39)
The influence of non-linear elastic systems on the measuring precision of measuring and control systems for moments/forces in static mode

The strength condition imposed in both linear deformations and in the superior order deformations is:

The measure variations $y_{\text{max}}$, $y_{\text{maxL}}$, $l_m$, $l_{\text{maxL}}$, $\beta$, $\beta_{\text{L}}$ are presented in the following diagrams:

\[
y_{\text{maxL}} = \frac{c_M}{54} \cdot l \tag{40}
\]

\[
l_{\text{mL}} = \frac{2}{3} \cdot l \tag{41}
\]

\[
\alpha_L = \frac{c_M}{12} \tag{42}
\]

\[
|\beta_L| = \frac{c_M}{4} \tag{43}
\]

\[
\sigma_{\text{max}} = \frac{M_{\text{max}}}{W} \tag{44}
\]

\[
\text{For } M_{\text{max}} = \frac{E}{l}, W = \frac{2 \cdot l}{b},
\]

\[
\text{it results }
\]

\[
\sigma_{\text{max}} = \frac{c_M \cdot E}{W \cdot 2 \cdot l} = \frac{c_M \cdot E}{2 \cdot l} \tag{45}
\]
2. Numeric example

We will take into consideration a flexible elastic element with the following dimensions:
- Thickness \( h = 0.1 \) [mm]
- Width \( b = 6 \) [mm]
- Length \( l = 50 \) [mm]
- Dimensions \( L = 150 \) [mm]
- \( I = 0.5 \times 10^{-3} \) [mm]
- \( E_0 \) – is Young’s modulus of beam material = \( 2.06 \times 10^5 \) [N/mm]

The action moment

\[
M_a = c_M \cdot \frac{E \cdot I}{l}
\]

(46)

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Diagram 3 – \( \beta, \beta_L \) variation

We get the value of the force

\[
F = \frac{M_t}{L}
\]

(48)

Table 1

<table>
<thead>
<tr>
<th>( c_M )</th>
<th>( \beta ) [degrees]</th>
<th>( \beta_L ) [degrees]</th>
<th>( M_t ) [N∙mm]</th>
<th>( F ) [N]</th>
</tr>
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</table>
The influence of non-linear elastic systems on the measuring precision of measuring and control systems for moments/forces in static mode

For
\[ S_a = \frac{\beta}{M_c} = \frac{\beta}{4 \cdot M_c} = \frac{\beta}{4 \cdot c_M \cdot \frac{E \cdot l}{I}} \text{ [radians]} \]

and
\[ S_{aL} = \frac{S_m}{M_c} = \frac{S_m}{4 \cdot c_M \cdot \frac{E \cdot l}{I}} = \frac{1}{16 \cdot c_M \cdot \frac{E \cdot l}{I}} \text{ [radians]} \]

we obtain the following values of \( S_a \) and \( S_{aL} \) depending on the \( c_M \) parameter in Table 2:

<table>
<thead>
<tr>
<th>( c_M )</th>
<th>( S_a )</th>
<th>( S_{aL} )</th>
<th>( R = \frac{S_a - S_{aL}}{S_{aL}} \cdot 100 % )</th>
<th>( \sigma_{max} [N/mm^2] )</th>
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</table>

\( \sigma_{max} \leq \sigma_a = 750 \cdots 800 [N/mm^2] \)

R – reprezintă creşterea în procente sensibilităţii sistemului de măsură în condiţii neliniare faţă de cea liniară.

3. Observation

All measures calculated in case of a linear elastic system are higher than the values determined for linear deformations.

For a difference between \( S_a \) and \( S_{aL} \) less then max. 1.25%, we can use a graduated linear scale.

4. Conclusions

- The advantage of using elastic slide consists in: as a simple design, use of a small volume of material in comparison with other elastic elements used for the same purpose, ease of calculation and dimensioning geometric characteristics and a high sensitivity of the measurement systems.

- In case of nonlinear deformations - up to the limit of acceptable tension of elastic slides – the graduated scale of the measuring system can be calibrated so as to show any amount of torque, thereby increasing measurement interval around 100%.

5. References


