SECTIONAL FORCES DIAGRAMS IN POLAR COORDINATES FOR CIRCULAR CANTILEVERS USING MATHCAD (II)

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Abstract - The sectional forces diagrams in polar coordinates for circular beams, loaded perpendicularly to their plane can be plotted using Mathcad (2011). The suggested method has the advantage of facilitating the verification of the sections where concentrated loads act (forces or moments) and allowing the identification of critical sections for bending or torsion. The method is based on the step-function from Mathcad, which provides an uniform definition of the functions of sectional forces (shear force, torsion and bending moments). The present paper aims to explain the method of determining these functions and show some numerical results obtained for three particular cases of a circular cantilever loaded perpendicularly to its plane.

Keywords: Diagrams, polar coordinates, Mathcad, circular cantilever

1. Problem Definition

The beam AB (Figure 1) is defined by a circular geometric axis is a circular cantilever with its free end in section A and its fixed end in section B. The central angle \( \alpha \) is variable. The vertical load \( q \) acts perpendicular to the plane of the cantilever (XOY) and is uniformly distributed along the sector \( AE \) (having a variable central angle \( \beta \) with respect to A). Additionally a vertical concentrated force \( P \) acts in section \( D \) (with a variable central angle \( \varphi \)), a bending moment \( M_i \) acts in section \( G \) (with a variable central angle \( \varphi \)) and a torsion moment \( M_t \) is applied in section \( H \) (with a variable central angle \( \gamma \)).

For this situation the following tasks are requested:
1. Find the general expressions of the shear forces \( T(\theta) \), torsion moments \( M_t(\theta) \) and bending moments \( M_i(\theta) \) functions of the uniformly distributed vertical load \( q \).
2. Find the equivalent force-couple system in section E corresponding to the exterior loads. Determine the general expressions of the reaction forces in the fixed support B.
3. Plot the diagrams of shear forces \( T(\theta) \), torsion moments \( M_t(\theta) \) and bending moments \( M_i(\theta) \), using the step-function in Mathcad.

Numerical values of the parameters:
Particular case A: \( R=1 \) m; \( P=1 \) kN; \( M_{i0}=1 \) kNm; \( M_{t0}=1 \) kNm; \( q=1 \) kN/m; \( \alpha=3\pi/2; \beta=5\pi/4; \varphi=3\pi/4; \gamma=3\pi/4 \).
Particular case B: \( R=100 \) mm; \( P=1 \) kN; \( M_{i0}=0; M_{t0}=0; q=0; \alpha=2\pi; \varphi=0 \).
Particular case C: \( R=100 \) mm; \( P=1 \) kN; \( M_{i0}=1 \) kNm; \( q=0; \alpha=2\pi; \varphi=0; \gamma=0 \)

2. General Analytical Functions of the Shear Forces, Torsion and Bending Moments

The general functions of the sectional forces caused by the uniformly distributed vertical load \( q \) can be determined by considering a beam element of length \( ds \) located at an angular distance \( \theta \) from the free end of the cantilever (Figure 2).

The corresponding elementary force will be [1]:

\[
dF=q \cdot ds = q \cdot R \cdot d\theta \quad (1)
\]

Figure 1: General 3D layout of the circular cantilever.
The analytical expressions of the shear force \( T(\theta) \) due to the uniformly distributed vertical load \( q \) on the sector \( AE \) will be obtained by integrating the elementary force \( dF \) and considering the sign convention for sectional forces (Figure 2):

\[
T(\theta) = \int_{0}^{\theta} (qRd\alpha) = qR \cdot \theta; \quad (2)
\]

By calculating the moment of the elementary force \( dF \) with respect to the normal axis \( O \) and the tangential axis \( t \) assigned to the current section and integrating on the arc length of angle \( \theta \) (Figure 2), the general expressions of the bending moment \( M_i(\theta) \) and torsional moment \( M_i(\theta) \) will be determined [3- Marin C, 2012]

\[
\begin{align*}
M_i(\theta) &= -\int_{0}^{\theta} (qR \cdot d\alpha) \cdot R \cdot \sin(\theta - \alpha) = -qR^2 \cdot (1 - \cos \theta); \\
M_i(\theta) &= -\int_{0}^{\theta} (qR \cdot d\alpha) \cdot R \cdot [1 - \cos(\theta - \alpha)] = -qR^2 \cdot (\theta - \sin \theta)
\end{align*}
\]

The equivalent force-couple system in section \( E \) corresponding to the uniformly distributed load \( q \) is composing of the sectional shear force (2) and the bending and torsional moments (3) calculated for the particular value of the angle \( \theta=\beta \) (Figure 3) [1- Marin C, 2006]:

\[
\begin{align*}
T_\beta &= qR \cdot \beta \\
M_{i\beta} &= -qR^2 \cdot (1 - \cos \beta) \\
M_{i\beta} &= -qR^2 \cdot (\beta - \sin \beta)
\end{align*}
\]

The reaction forces in the fixed support from section \( B \) \( (V_B, M_{iB} \text{ and } M_{iB}) \) will be determined using the equivalent force-couple system from section \( E \), the concentrated load \( P \) and the moments \( M_{i\theta} \text{ and } M_{i\theta} \) (Figure 3).

\[
\begin{align*}
V_B &= qR \cdot \beta - P \\
M_{iB} &= M_{i\theta} \cdot \sin \psi - M_{i\theta} \cdot \cos \gamma + PR \cdot \cos \phi + qR^2 \cdot (\beta \cdot \cos \beta - \sin \beta) \\
M_{iB} &= M_{i\theta} \cdot \cos \psi + M_{i\theta} \cdot \sin \gamma - PR \cdot (1 + \sin \phi) + qR^2 \cdot (1 - \beta \cdot \sin \beta - \cos \beta)
\end{align*}
\]

3. Sectional Forces Diagrams in Polar Coordinates

The diagrams of sectional forces in polar coordinates will be plotted using the step-function available in Mathcad [5- Mathcad, 2011]. Their geometrical axis will be a circle with radius \( R=10 \ R \).

The same sign convention as in the case of straight beams will be adopted.
\[ T(\theta) = 10 \cdot R + (qR \cdot \theta)(\Phi(\theta) - \Phi(\theta - \beta)) + T_{p}(\Phi(\theta - \beta) - \Phi(\theta - \alpha)) + P \cdot (\Phi(\theta - \varphi) - \Phi(\theta - \alpha)) \]

\[ M_{i}(\theta) = 10 \cdot PR - qR^{2} \cdot (1 - \cos \theta) \cdot (\Phi(\theta) - \Phi(\theta - \beta)) + PR \cdot \sin(\theta - \varphi) \cdot (\Phi(\theta - \varphi) - \Phi(\theta - \alpha)) + \]
\[ \quad + (M_{i_{0}} \cdot \cos(\theta - \beta) - M_{i_{0}} \sin(\theta - \beta)) \cdot (\Phi(\theta - \beta) - \Phi(\theta - \alpha)) + \]
\[ \quad + M_{i_{0}} \cdot \cos(\theta - \varphi) \cdot (\Phi(\theta - \varphi) - \Phi(\theta - \alpha)) - M_{i_{0}} \cdot \sin(\theta - \gamma) \cdot (\Phi(\theta - \gamma) - \Phi(\theta - \alpha)) \]
\[ M_{t}(\theta) = 10 \cdot PR - qR^{2} \cdot (\theta - \sin \theta) \cdot (\Phi(\theta) - \Phi(\theta - \beta)) + PR \cdot (1 - \cos(\theta - \varphi)) \cdot (\Phi(\theta - \varphi) - \Phi(\theta - \alpha)) + \]
\[ \quad + (M_{i_{0}} \cdot \sin(\theta - \beta) + M_{i_{0}} \cos(\theta - \beta)) \cdot (\Phi(\theta - \beta) - \Phi(\theta - \alpha)) + \]
\[ \quad + M_{i_{0}} \cdot \sin(\theta - \varphi) \cdot (\Phi(\theta - \varphi) - \Phi(\theta - \alpha)) + M_{i_{0}} \cdot \cos(\theta - \gamma) \cdot (\Phi(\theta - \gamma) - \Phi(\theta - \alpha)) \]

(6)

4. Diagrams of Sectional Forces

4.1. Particular Case A

Using the numerical values for the case A: \( R=1 \, m; \)
\( P=1 \, kN; \) \( M_{i_{0}}=1 \, kNm; \) \( M_{t_{0}}=1 \, kNm; \) \( q=1 \, kN/m; \) \( \alpha=\frac{\pi}{2}; \)
\( \beta=\frac{5\pi}{4}; \) \( \varphi=3\pi/4; \) \( \gamma=3\pi/4, \) the following reactions in the fixed support \( B \) will be obtained:

\[ H_{B} = 2,927 \, kN \]
\[ M_{i_{B}} = -1,045 \, kNm \]
\[ M_{t_{B}} = 2,777 \, kNm \]

The sectional forces diagrams (Figure 4 – 6) were plotted using the relations (6) in Mathcad [5-Mathcad, 2011].

Figure 4: Shear forces diagram – case A

Figure 5: Bending moments diagram – case A

Figure 6: Torsional moments diagram – case A
Using the above mentioned diagrams, the maximum values of the sectional forces can be determined as well as the location of the critical sections, corresponding to $M_{i\text{max}}$ and $M_{t\text{max}}$:

- The maximum shear force (Figure 4) located at $\theta=\frac{3\pi}{2}$ is $T_{\text{max}}=2,927$ kN.
- The maximum bending moment (Figure 5) located at $\theta=\frac{5\pi}{4}$ is $M_{i\text{max}}=-1,932$ kNm.
- The maximum bending moment (Figure 6) located at $\theta=\frac{3\pi}{2}$ is $M_{t\text{max}}=-2,777$ kNm.

4.2. Particular Case B

Using the numerical values for the case B: $R=100$ mm; $P=1$ kN; $M_{\alpha}=0$; $M_{\phi}=0$; $q=0$; $\alpha=2\pi$; $\varphi=0$, the following reactions in the fixed support $B$ will be obtained:

$$
\begin{align*}
H_B &= 1 \text{ kN} \\
M_{iB} &= 0 \\
M_{tB} &= 0
\end{align*}
$$

The sectional forces diagrams (Figure 7 – 9) were plotted using the relations (6) in Mathcad [5-Mathcad,2011].

4.3. Particular Case C

Using the numerical values for the case C: $R=100$ mm; $P=1$ kN; $M_{\alpha}=0$; $M_{\phi}=1$ kNm; $q=0$; $\alpha=2\pi$; $\varphi=0$, $\gamma=0$, the following reactions in the fixed support $B$ will be obtained:

$$
\begin{align*}
H_B &= 1 \text{ kN} \\
M_{iB} &= 0 \\
M_{tB} &= 0
\end{align*}
$$

The sectional forces diagrams (Figure 10 – 12) were plotted using the relations (6) in Mathcad [5-Mathcad,2011].
Sectional forces diagrams in polar coordinates for circular cantilevers using Mathcad (II)

Figure 10: Shear forces diagram – case C

Figure 11: Bending moments diagram – case C

Figure 12: Torsional moments diagram – case C

4. Conclusions

Using the results above the following conclusions can be drawn:

- The method presented above allows the fast determination of the reaction forces depending on the input parameters. Plotting of the internal forces diagrams and the visualization of maximal and minimal values are enhanced.
- The diagrams in polar coordinates allow the identification of the location of critical sections, together with the corresponding maximum values of sectional forces.
- The particular case A is a more general one and it confirms the presence of jumps in the diagrams in the sections of concentrated loads. The reactions in the fixed support read from the diagrams are the same with the values determined by means of relations (4).
- The particular case B, where the cantilever is a full circle loaded in section A with a concentrated load \( P \) shows a constant shear force along the entire length of the beam. In the middle \( (\theta=\pi) \) the bending moments are null and the torsional moments are maximum. These results are in perfect agreement with observed experimental data.
- The particular case C corresponds to the practical situation of a cylindrical helical spring with a very small coil angle. The sectional forces diagrams show constant shear forces and torsional moments along the entire length of the beam and no bending moments. This is again in very good agreement with observed experimental data.
- The method has a high general character and can be verified using already existing experimental data.

5. References